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The standard model à la Connes–Lott

Daniel Kastler [1], Thomas Schücker *

CPT, CNRS – Luminy, Case 907, 13288 Marseille, Cedex 9, France

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Abstract

The relations among coupling constants and masses in the standard model à la Connes–Lott with general scalar product are computed in detail. We find a relation between the top and the Higgs masses. For $m_t = 174 \pm 22$ GeV it yields $m_H = 277 \pm 40$ GeV. The Connes–Lott theory privileges the masses $m_t = 160.4$ GeV and $m_H = 251.8$ GeV.

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0. Introduction

By now the standard model of electro-weak and strong interactions in the setting of non-commutative geometry [1] is well documented [1–6] and needs no further introduction. The main virtue of non-commutative geometry in the context of a Yang–Mills–Higgs model is that the entire Higgs sector including the choice of the scalar representation has one common geometrical origin and is not just added by hand.

The action of a Yang–Mills–Higgs model consists of five pieces: the Yang–Mills action, the Klein–Gordon action, the integrated Higgs potential, the Dirac action, and the Yukawa couplings. Of these only the first has a genuine geometrical interpretation. As a consequence, the representation of the gauge potentials is not arbitrary, it is the adjoint representation. Also, the cubic and quartic self-couplings of the gauge potentials are not arbitrary, they are computed from the gauge invariant scalar product on the Lie algebra and the structure

* Corresponding author. E-mail: schucker@cpt.univ.mrs.fr. Also at Université de Provence.
[1] Also at Université d'Aix-Marseille II.

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constants. Similarly, the couplings of the gauge potentials to the scalars and fermions in the Klein–Gordon and Dirac actions are fixed by geometry, as 'minimal couplings'. All the other coupling constants, the quadratic, cubic, and quartic ones in the Higgs potential, and the trilinear Yukawa couplings are arbitrary except for gauge invariance. Also, the scalar and the left- and right-handed fermion representations are arbitrary.

The action of a Connes–Lott model consists of only two pieces, the non-commutative Yang–Mills and Dirac actions. When expanded in terms of ordinary fields the non-commutative gauge potential consists of the ordinary gauge potential in the adjoint representation and a scalar field in a representation, that is computed. At the same time, the non-commutative Yang–Mills action yields the ordinary Yang–Mills action, the Klein–Gordon action and the Higgs potential. Just as the self-couplings of the gauge potentials, the self-couplings of the scalars are now computed from an invariant scalar product in the non-commutative sense and the underlying algebraic structure. Finally the non-commutative Dirac action produces the ordinary Dirac action and the Yukawa couplings. Input of a Connes–Lott model is a finite dimensional involutive algebra, the two fermion representations and their mass matrix. These data then produce a very particular Yang–Mills–Higgs model [7]. Its gauge group is the group of unitary elements in the algebra or a subgroup thereof. This model features constraint gauge couplings and a fixed scalar representation. Its gauge and scalar boson masses are determined in terms of the fermion masses. For the standard model, the scalar representation comes out to be a weak isospin doublet and with the simplest scalar product one has [3,4,8]

$$g_3 = g_2, \qquad \sin^2\theta_{\mathrm{w}} = \tfrac{3}{8}, \qquad m_{\mathrm{t}} = 2\, m_{\mathrm{W}}, \qquad m_{\mathrm{H}} = 3.14\, m_{\mathrm{W}}. \tag{1}$$

All four relations are unstable under quantum corrections [9] and raise the question of how to quantize a field theory of non-commutative geometry. If interpreted at their natural scale m_W, the first two relations are in contradiction with experiment, the third is close to the recently announced top mass [10]. When calculating with a more general scalar product [3] one still gets a relation among coupling constants,

$$\frac{1}{3}\frac{1}{\alpha_3} + \frac{0.25 - \sin^2\theta_{\mathrm{w}}}{\alpha_{\mathrm{em}}} = 0, \tag{2}$$

and a conflict with experiment. Connes and Lott [8] also wrote down the most general gauge invariant scalar product. Due to the high degree of reducibility of the fermion representations in the standard model, the general scalar product destroys all four relations, however leaving a relation between the top and the Higgs masses and leaving an inequality for $\sin^2\theta_{\mathrm{w}}$. There is a natural subclass of scalar products, that determines the top and Higgs masses as in Eqs. (1). Alvarez et al. [11] have carried out a renormalization group analysis of these two mass relations in ordinary quantum field theory. They find a weak scale dependence only.

The purpose of this article is to give the computational details of the standard model with general scalar product and to discuss the phenomenological implications. The more mathematically inclined reader is referred to a companion paper [12].

1. Input of the standard model in the Connes–Lott scheme

The standard model in non-commutative geometry is described by two real algebras, one for electro-weak interactions: $\mathcal{A} := \mathbb{H} \oplus \mathbb{C}$ with group of unitaries $SU(2) \times U(1)$, and one for strong interactions: $\mathcal{A}' := M_3(\mathbb{C}) \oplus \mathbb{C}$ with group of unitaries $U(3) \times U(1)$. We denote by $\mathbb{H}$ the algebra of quaternions. Its elements are complex 2×2 matrices of the form

$$\begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix}, \quad x, y \in \mathbb{C}. \tag{3}$$

Both algebras $\mathcal{A}$ and $\mathcal{A}'$ are represented on the same Hilbert space $\mathcal{H} = \mathcal{H}_{\mathrm{L}} \oplus \mathcal{H}_{\mathrm{R}}$ of left-and right-handed fermions,

$$\mathcal{H}_{\mathrm{L}} = (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \ \oplus \ (\mathbb{C}^2 \otimes \mathbb{C}^N), \tag{4}$$

$$\mathcal{H}_{\mathrm{R}} = ((\mathbb{C} \oplus \mathbb{C}) \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \ \oplus \ (\mathbb{C} \otimes \mathbb{C}^N). \tag{5}$$

The first factor denotes weak isospin, the second N generations, $N = 3$, and the third denotes colour triplets and singlets. With respect to the standard basis

$$\begin{pmatrix} u \\ d \end{pmatrix}_{\mathrm{L}}, \begin{pmatrix} c \\ s \end{pmatrix}_{\mathrm{L}}, \begin{pmatrix} t \\ b \end{pmatrix}_{\mathrm{L}}, \begin{pmatrix} \nu_e \\ e \end{pmatrix}_{\mathrm{L}}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_{\mathrm{L}}, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_{\mathrm{L}} \tag{6}$$

of $\mathcal{H}_{\mathrm{L}}$ and

$$\begin{matrix} u_{\mathrm{R}}, & c_{\mathrm{R}}, & t_{\mathrm{R}}, \\ d_{\mathrm{R}}, & s_{\mathrm{R}}, & b_{\mathrm{R}}, \end{matrix} \quad e_{\mathrm{R}}, \quad \mu_{\mathrm{R}}, \quad \tau_{\mathrm{R}} \tag{7}$$

of $\mathcal{H}_{\mathrm{R}}$, the representations are given by block diagonal matrices. For $(a, b) \in \mathbb{H} \oplus \mathbb{C}$ we set

$$B := \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix} \tag{8}$$

and define a representation of $\mathcal{A}$ by

$$\rho(a, b) := \begin{pmatrix} a \otimes 1_N \otimes 1_3 & 0 & 0 & 0 \\ 0 & a \otimes 1_N & 0 & 0 \\ 0 & 0 & B \otimes 1_N \otimes 1_3 & 0 \\ 0 & 0 & 0 & \bar{b} 1_N \end{pmatrix} = \begin{pmatrix} \rho_{\mathrm{L}}(a) & 0 \\ 0 & \rho_{\mathrm{R}}(b) \end{pmatrix} \tag{9}$$

and for $(c, d) \in M_3(\mathbb{C}) \oplus \mathbb{C}$ we define an $\mathcal{A}'$ representation

$$\rho'(c, d) := \begin{pmatrix} 1_2 \otimes 1_N \otimes c & 0 & 0 & 0 \\ 0 & d 1_2 \otimes 1_N & 0 & 0 \\ 0 & 0 & 1_2 \otimes 1_N \otimes c & 0 \\ 0 & 0 & 0 & d 1_N \end{pmatrix}. \tag{10}$$

The last piece of input is the fermion mass matrix $\mathcal{M}$ which constitutes the self-adjoint 'internal Dirac operator':

$$\mathcal{D} := \begin{pmatrix} 0 & 0 & \begin{pmatrix} M_u \otimes 1_3 & 0 \\ 0 & M_d \otimes 1_3 \end{pmatrix} & 0 \\ 0 & 0 & 0 & \begin{pmatrix} 0 \\ M_e \end{pmatrix} \\ \begin{pmatrix} M_u^* \otimes 1_3 & 0 \\ 0 & M_d^* \otimes 1_3 \end{pmatrix} & 0 & 0 & 0 \\ 0 & (0 \quad M_e^*) & 0 & 0 \end{pmatrix} =: \begin{pmatrix} 0 & \mathcal{M} \\ \mathcal{M}^* & 0 \end{pmatrix} \tag{11}$$

with

$$M_u := \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \quad M_d := C_{\mathrm{KM}} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix},$$
$$M_e := \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}, \tag{12}$$

where C_{KM} denotes the Cabbibo–Kobayashi–Maskawa matrix. All indicated fermion masses are supposed positive and different. Note that the strong interactions are vector-like: the chirality operator

$$\chi = \begin{pmatrix} -1_2 \otimes 1_N \otimes 1_3 & 0 & 0 & 0 \\ 0 & -1_2 \otimes 1_N & 0 & 0 \\ 0 & 0 & 1_2 \otimes 1_N \otimes 1_3 & 0 \\ 0 & 0 & 0 & 1_N \end{pmatrix} \tag{13}$$

and the 'Dirac operator' commute with $\mathcal{A}'$

$$[\mathcal{D}, \rho'(\mathcal{A}')] = 0, \tag{14}$$

$$[\chi, \rho'(\mathcal{A}')] = 0. \tag{15}$$

2. Connes–Lott model building kit, internal space

With this input – an involution algebra $\mathcal{A}$, a faithful representation ρ of $\mathcal{A}$ on $\mathcal{H}$, that decomposes into a left-handed representation ρ_{L} on $\mathcal{H}_{\mathrm{L}}$ and a right-handed one, and a 'Dirac operator' $\mathcal{D}$ – Connes constructs the central piece of his model building kit, a differential algebra $\Omega_{\mathcal{D}}\mathcal{A}$. This construction may seem complicated at first sight, but it has profound roots in non-commutative geometry.

It starts with an auxiliary differential algebra $\Omega\mathcal{A}$, the so-called universal differential envelope of $\mathcal{A}$:

$$\Omega^0\mathcal{A} := \mathcal{A}, \tag{16}$$

$\Omega^1\mathcal{A}$ is generated by symbols δa, $a \in \mathcal{A}$, with relations

$$\delta 1 = 0, \tag{17}$$

$$\delta(aa') = (\delta a)a' + a\delta a'. \tag{18}$$

For the moment $\mathcal{A}$ is an arbitrary involution algebra with generic elements a, a', ... Forget about quaternions and the second algebra $\mathcal{A}'$. $\Omega^1\mathcal{A}$ consists of finite sums of terms of the form $a_0\delta a_1$,

$$\Omega^1\mathcal{A} = \left\{ \sum_j a_0^j \delta a_1^j,\ a_0^j, a_1^j \in \mathcal{A} \right\} \tag{19}$$

and likewise for higher p

$$\Omega^p\mathcal{A} = \left\{ \sum_j a_0^j \delta a_1^j \cdots \delta a_p^j,\ a_q^j \in \mathcal{A} \right\}. \tag{20}$$

The differential δ is defined by

$$\delta(a_0\delta a_1 \cdots \delta a_p) := \delta a_0 \delta a_1 \cdots \delta a_p. \tag{21}$$

The involution * is extended from the algebra $\mathcal{A}$ to $\Omega^1\mathcal{A}$ by putting

$$(\delta a)^* := \delta(a^*) =: \delta a^*. \tag{22}$$

With the definition

$$(\varphi\psi)^* = \psi^*\varphi^*, \tag{23}$$

the involution is extended to the whole differential envelope.

The next step is to extend the representation $\rho := \rho_{\mathrm{L}} \oplus \rho_{\mathrm{R}}$ on $\mathcal{H} := \mathcal{H}_{\mathrm{L}} \oplus \mathcal{H}_{\mathrm{R}}$ from the algebra $\mathcal{A}$ to its universal differential envelope $\Omega\mathcal{A}$. This extension deserves a new name:

$$\pi : \Omega\mathcal{A} \longrightarrow \bigoplus_p \mathrm{End}(\mathcal{H}),$$
$$\pi(a_0\delta a_1 \cdots \delta a_p) := (-\mathrm{i})^p \rho(a_0)[\mathcal{D}, \rho(a_1)] \cdots [\mathcal{D}, \rho(a_p)]. \tag{24}$$

A straightforward calculation shows that π is in fact a representation of $\Omega\mathcal{A}$ as graded involution algebra, and we are tempted to define also a differential, again denoted by δ, on $\pi(\Omega\mathcal{A})$ by

$$\delta\pi(\hat{\varphi}) := \pi(\delta\hat{\varphi}). \tag{25}$$

However, this definition does not make sense if there are forms $\hat{\varphi} \in \Omega\mathcal{A}$ with $\pi(\hat{\varphi}) = 0$ and $\pi(\delta\hat{\varphi}) \neq 0$. By dividing out these unpleasant forms, we finally arrive at the differential algebra $\Omega_{\mathcal{D}}\mathcal{A}$, the real thing

$$\Omega_{\mathcal{D}}\mathcal{A} := \frac{\pi\ (\Omega\mathcal{A})}{J} \tag{26}$$

with

$$J := \pi\,(\delta \ker \pi) =: \bigoplus_p J^p \tag{27}$$

(J for junk). On the quotient now, the differential (25) is well defined. Degree by degree we have

$$\Omega^0_{\mathcal{D}}\mathcal{A} = \rho(\mathcal{A}) \tag{28}$$

because $J^0 = 0$,

$$\Omega^1_{\mathcal{D}}\mathcal{A} = \pi(\Omega^1\mathcal{A}) \tag{29}$$

because ρ is faithful, and in degree $p \geq 2$

$$\Omega^p_{\mathcal{D}}\mathcal{A} = \frac{\pi(\Omega^p\mathcal{A})}{\pi(\delta(\ker\pi)^{p-1})}. \tag{30}$$

While $\Omega\mathcal{A}$ has no cohomology, $\Omega_{\mathcal{D}}\mathcal{A}$ in general has. In fact, in infinite dimensions, if $\mathcal{F}$ is the algebra of complex functions on space–time M represented on the square integrable spinors by multiplication and if $\mathcal{D}$ is the genuine Dirac operator then $\Omega_{\mathcal{D}}\mathcal{F}$ is de Rham's differential algebra of differential forms on M.

We come back to our finite-dimensional case. Recall that the elements of the auxiliary differential algebra $\Omega\mathcal{A}$ that we introduced for book keeping purposes only are abstract entities defined in terms of symbols and relations. On the other hand the elements of $\Omega_{\mathcal{D}}\mathcal{A}$, the 'forms' are operators on the Hilbert space $\mathcal{H}$, i.e. concrete matrices of complex numbers. Therefore there is a natural scalar product defined by

$$\langle\hat\varphi, \hat\psi\rangle := \mathrm{tr}(\hat\varphi^*\hat\psi), \quad \hat\varphi, \hat\psi \in \pi(\Omega^p\mathcal{A}), \tag{31}$$

for elements of equal degree and by zero for two elements of different degree. With this scalar product $\Omega_{\mathcal{D}}\mathcal{A}$ is a subspace of $\pi(\Omega\mathcal{A})$ by definition orthogonal to the junk. As a subspace $\Omega_{\mathcal{D}}\mathcal{A}$ inherits a scalar product which deserves a special name (,). It is given by

$$(\varphi, \psi) = \langle\hat\varphi, P\hat\psi\rangle, \quad \varphi, \psi \in \Omega^p_{\mathcal{D}}\mathcal{A}, \tag{32}$$

where P is the orthogonal projector in $\pi(\Omega\mathcal{A})$ onto the ortho-complement of J, and $\hat\varphi$ and $\hat\psi$ are any representatives in their classes. Again the scalar product vanishes for forms with different degree. For real algebras all traces must be understood as real part of the trace.

Now suppose that the left and right representations are reducible as the case in the standard model. Then there is an obvious generalization of the scalar product (31). It is constructed by taking the trace over each irreducible part of $\mathcal{H}$ separately and by multiplying each trace by an independent positive constant. The most general scalar product in this context reads [8]

$$\langle\hat\varphi, \hat\psi\rangle := \mathrm{tr}(\hat\varphi^*\hat\psi z), \quad \hat\varphi, \hat\psi \in \pi(\Omega^p\mathcal{A}), \tag{33}$$

z is any positive operator on $\mathcal{H}$, that commutes with $\rho(\mathcal{A})$ and with $\mathcal{D}$.

Let us remark the existence of a natural subclass of scalar products [8] defined by elements z, that are not only in the commutant of $\mathcal{A}$ but are taken from the image under ρ of the centre of $\mathcal{A}$.

At this stage, there is a first contact with gauge theories. Consider the vector space of anti-Hermitian 1-forms

$$\{H \in \Omega^1_{\mathcal{D}}\mathcal{A},\ H^* = -H\}. \tag{34}$$

Let us call these elements Higgses. The space of Higgses carries an affine representation of the group of unitaries

$$G = \{g \in \mathcal{A},\ gg^* = g^*g = 1\} \tag{35}$$

defined by

$$\begin{aligned} H^g &:= \rho(g)H\rho(g^{-1}) + \rho(g)\delta(\rho(g^{-1})) \\ &= \rho(g)H\rho(g^{-1}) + (-\mathrm{i})\rho(g)[\mathcal{D}, \rho(g^{-1})] \\ &= \rho(g)[H - \mathrm{i}\,\mathcal{D}]\rho(g^{-1}) + \mathrm{i}\,\mathcal{D}. \end{aligned} \tag{36}$$

H^g is the 'gauge transform of H'. This transformation law makes the Higgs play the role of a (finite-dimensional) gauge potential. In fact every H defines a covariant derivative $\delta + H$, covariant under the left action of G on $\Omega_{\mathcal{D}}\mathcal{A}$:

$${}^g\psi := \rho(g)\psi, \quad \psi \in \Omega_{\mathcal{D}}\mathcal{A}, \tag{37}$$

which means

$$(\delta + H^g)\,{}^g\psi = {}^g\,[(\delta + H)\psi]. \tag{38}$$

Also we define the curvature C of H by

$$C := \delta H + H^2 \in \Omega^2_{\mathcal{D}}\mathcal{A}. \tag{39}$$

Note that here and later H^2 is considered as an element of $\Omega^2_{\mathcal{D}}\mathcal{A}$ which means it is the projection P applied to $H^2 \in \pi(\Omega^2\mathcal{A})$. The curvature C is a Hermitian 2-form with *homogeneous* gauge transformations

$$C^g := \delta(H^g) + (H^g)^2 = \rho(g)C\rho(g^{-1}). \tag{40}$$

Finally we define the preliminary Higgs potential $V_0(H)$, a functional on the space of Higgses, by

$$V_0(H) := (C, C) = \mathrm{tr}[(\delta H + H^2)P(\delta H + H^2)]. \tag{41}$$

It is a polynomial of degree 4 in H with real, non-negative values. Furthermore it is gauge invariant, $V_0(H^g) = V_0(H)$, because of the homogeneous transformation property of the curvature C and because the orthogonal projector P commutes with all gauge transformations, $\rho(g)P = P\rho(g)$. The most remarkable property of the preliminary Higgs potential is that, in most cases, its minimum spontaneously breaks the group G. To see this, we introduce the change of variables

$$\Phi := H - \mathrm{i}\mathcal{D}. \tag{42}$$

This variable transforms homogeneously:

$$\Phi^g = H^g - \mathrm{i}\mathcal{D} = \rho(g)[H - \mathrm{i}\mathcal{D}]\rho(g^{-1}) + \mathrm{i}\mathcal{D} - \mathrm{i}\mathcal{D} = \rho(g)\Phi\rho(g^{-1}). \tag{43}$$

Now $H = 0$, or equivalently $\Phi = -\mathrm{i}\mathcal{D}$, is certainly a minimum of the preliminary Higgs potential and this minimum spontaneously breaks G if it is gauge variant. For instance, for vector-like models like electro- and chromo-dynamics, H vanishes identically and the gauge bosons remain massless.

The invariance group of the Higgs potential is the group of unitaries G, a subset of the algebra $\mathcal{A}$. G can be reduced to a special subgroup by means of a so-called unimodularity condition. These conditions are defined on G^0, the connected component of the identity in G. For a finite-dimensional algebra $\mathcal{A}$ represented on a finite-dimensional Hilbert space $\mathcal{H}$, the unimodularity conditions take a simple form. Every element $g \in G^0$ can be written

$$g = \mathrm{e}^X, \tag{44}$$

where X is an element in the Lie algebra $\mathfrak{g}$ of G. The Lie algebra $\mathfrak{g}$ is again a subset of the algebra $\mathcal{A}$,

$$\mathfrak{g} = \{X \in \mathcal{A},\ X^* + X = 0\}. \tag{45}$$

Choose an element p in the centre of $\mathcal{A}$ such that $\operatorname{tr}\rho(p) \in \mathbb{Z}$, p stands for projection. For every p, there is a unimodularity condition

$$\operatorname{tr}\rho(Xp) = 0 \tag{46}$$

defining a subgroup of G^0,

$$G_p := \{g = \mathrm{e}^X \in G^0,\ \operatorname{tr}\rho(Xp) = 0\}. \tag{47}$$

3. Internal space of the standard model

We now apply the construction outlined above to the standard model. Obviously, the standard model is not the right example to get familiar with the Connes–Lott scheme. Miraculously enough, the standard model contains the minimax example, analogue of the Georgi–Glashow $SO(3)$ model [13] in the Yang–Mills–Higgs scheme (a maximum of pleasure with a minimum of effort). This example represents the electro-weak algebra $\mathcal{A} = \mathbb{H} \oplus \mathbb{C}$ on *two* generations of leptons. Its only drawback is neutrinos with electric charge, a drawback that can be corrected by adding strong interactions.

Anyway, let us start the computation of the differential algebra $\Omega_{\mathcal{D}}\mathcal{A}$ for the electro-weak algebra with generic element $(a, b) \in \mathbb{H} \oplus \mathbb{C}$ represented on the long list of fermions. A general 1-form is a sum of terms

$$\pi((a_0, b_0)\delta(a_1, b_1)) = -\mathrm{i}\begin{pmatrix} 0 & \rho_L(a_0)\,(\mathcal{M}\rho_R(b_1) - \rho_L(a_1)\mathcal{M}) \\ \rho_R(b_0)\,(\mathcal{M}^*\rho_L(a_1) - \rho_R(b_1)\mathcal{M}^*) & 0 \end{pmatrix} \tag{48}$$

and as vector space

$$\Omega^1_{\mathcal{D}}\mathcal{A} = \left\{ \mathrm{i}\begin{pmatrix} 0 & \rho_L(h)\mathcal{M} \\ \mathcal{M}^*\rho_L(\tilde{h}^*) & 0 \end{pmatrix},\ h, \tilde{h} \in \mathbb{H} \right\}. \tag{49}$$

The Higgs being an anti-Hermitian 1-form

$$H = \mathrm{i}\begin{pmatrix} 0 & \rho_L(h)\mathcal{M} \\ \mathcal{M}^*\rho_L(h^*) & 0 \end{pmatrix}, \quad h = \begin{pmatrix} h_1 & -\bar{h}_2 \\ h_2 & \bar{h}_1 \end{pmatrix} \in \mathbb{H}, \tag{50}$$

is parametrized by one complex doublet

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad h_1, h_2 \in \mathbb{C}. \tag{51}$$

The junk in degree 2 turns out to be

$$J^2 = \left\{ \mathrm{i}\begin{pmatrix} j \otimes \Delta & 0 \\ 0 & 0 \end{pmatrix},\ j \in \mathbb{H} \right\} \tag{52}$$

with

$$\Delta := \frac{1}{2}\begin{pmatrix} (M_u M_u^* - M_d M_d^*) \otimes 1_3 & 0 \\ 0 & -M_e M_e^* \end{pmatrix}. \tag{53}$$

To project it out, we use the general scalar product (33) with the real part of the trace. Here the most general z, that commutes with $\rho(\mathcal{A})$ and $\mathcal{D}$, has the form

$$z = \begin{pmatrix} 1_2 \otimes 1_N \otimes x & 0 & 0 & 0 \\ 0 & 1_2 \otimes y & 0 & 0 \\ 0 & 0 & 1_2 \otimes 1_N \otimes x & 0 \\ 0 & 0 & 0 & y \end{pmatrix}, \tag{54}$$

where y is a positive, diagonal $N \times N$ matrix and x is a positive 3×3 matrix. Note that this z also commutes with the chirality operator χ. The scalar product defined with this z has a natural interpretation. Indeed, we shall see later that, without loss of generality, we may take x to be a positive multiple of the identity. Then, the general scalar product is just a sum of the simplest scalar products in each irreducible part of the fermion representation, each weighted with a separate positive constant. We have four irreducible parts, the three lepton families and all quarks together. Due to the Cabbibo–Kobayashi–Maskawa mixing, the ponderations of the three quark families are identical. If, in addition, we suppose that z lie in $\rho(\text{centre}\mathcal{A})$ then we have $x = \lambda 1_3$, $y = \lambda 1_N$ with a positive constant λ.

With respect to the general scalar product, we can write the 2-forms as

$$\Omega^2_{\mathcal{D}}\mathcal{A} = \left\{ \begin{pmatrix} \tilde{c} \otimes \Sigma & 0 \\ 0 & \mathcal{M}^*\rho_L(c)\mathcal{M} \end{pmatrix},\ \tilde{c}, c \in \mathbb{H} \right\} \tag{55}$$

with

$$\Sigma := \frac{1}{2}\begin{pmatrix} (M_u M_u^* + M_d M_d^*) \otimes 1_3 & 0 \\ 0 & M_e M_e^* \end{pmatrix}. \tag{56}$$

Since π is a homomorphism of involution algebras, the product in $\Omega_{\mathcal{D}}\mathcal{A}$ is given by matrix multiplication followed by the orthogonal projection P and the involution is given by transposition complex conjugation. In order to calculate the differential δ, we go back to the universal differential envelope. The result is

$$\begin{gathered} \delta : \Omega^1_{\mathcal{D}}\mathcal{A} \longrightarrow \Omega^2_{\mathcal{D}}\mathcal{A} \\ \mathrm{i}\begin{pmatrix} 0 & \rho_{\mathrm{L}}(h)\mathcal{M} \\ \mathcal{M}^*\rho_{\mathrm{L}}(\tilde{h}^*) & 0 \end{pmatrix} \longmapsto \begin{pmatrix} \tilde{c}\otimes\Sigma & 0 \\ 0 & \mathcal{M}^*\rho_{\mathrm{L}}(c)\mathcal{M} \end{pmatrix} \end{gathered} \tag{57}$$

with $\tilde{c} = c = h + \tilde{h}^*$.

We are now in position to compute the curvature and the preliminary Higgs potential:

$$C := \delta H + H^2 = (1 - |\varphi|^2)\begin{pmatrix} 1_2\otimes\Sigma & 0 \\ 0 & \mathcal{M}^*\mathcal{M} \end{pmatrix}, \tag{58}$$

where we have introduced the homogeneous scalar variable

$$\Phi := H - \mathrm{i}\mathcal{D} =: \mathrm{i}\begin{pmatrix} 0 & \rho_{\mathrm{L}}(\varphi)\mathcal{M} \\ \mathcal{M}^*\rho_{\mathrm{L}}(\varphi^*) & 0 \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_1 & -\bar{\varphi}_2 \\ \varphi_2 & \bar{\varphi}_1 \end{pmatrix} \in \mathbb{H}, \tag{59}$$

$$|\varphi|^2 := |\varphi_1|^2 + |\varphi_2|^2. \tag{60}$$

The preliminary Higgs potential

$$\begin{aligned} V_0 = \operatorname{tr}[C^2 z] = (1 - |\varphi|^2)^2 (&\tfrac{3}{2}\operatorname{tr}[(M_u^* M_u)^2]\operatorname{tr} x + \tfrac{3}{2}\operatorname{tr}[(M_d^* M_d)^2]\operatorname{tr} x \\ &+ \tfrac{1}{2}\operatorname{tr}[M_u M_u^* M_d M_d^*]\operatorname{tr} x + \tfrac{1}{2}\operatorname{tr}[M_d M_d^* M_u M_u^*]\operatorname{tr} x \\ &+ \tfrac{3}{2}\operatorname{tr}[(M_e^* M_e)^2 y]) \end{aligned} \tag{61}$$

breaks the $SU(2)\times U(1)$ symmetry down to $U(1)$.

Finally we must compute the differential algebra $\Omega_{\mathcal{D}}\mathcal{A}'$ of the strong algebra. As the strong interactions are vector-like this is trivial:

$$\Omega^0_{\mathcal{D}}\mathcal{A}' = \rho'(\mathcal{A}'), \qquad \Omega^p_{\mathcal{D}}\mathcal{A}' = 0, \quad p \geq 1. \tag{62}$$

Consequently there is no Higgs and no Higgs potential in the strong internal space. For later use, we still need the general positive operator z' on $\mathcal{H}$, that commutes with $\rho'(\mathcal{A}')$ and with the internal Dirac operator $\mathcal{D}$:

$$z' = \begin{pmatrix} \begin{pmatrix} r & 0 \\ 0 & C_{\mathrm{KM}}\, s\, C_{\mathrm{KM}}^* \end{pmatrix}\otimes 1_3 & 0 & \begin{pmatrix} k & 0 \\ 0 & C_{\mathrm{KM}}\, p \end{pmatrix}\otimes 1_3 & 0 \\ 0 & \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} & 0 & \begin{pmatrix} 0 \\ w \end{pmatrix} \\ \begin{pmatrix} k & 0 \\ 0 & p C_{\mathrm{KM}}^* \end{pmatrix}\otimes 1_3 & 0 & \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}\otimes 1_3 & 0 \\ 0 & (0 \quad w) & 0 & v \end{pmatrix}, \tag{63}$$

where r, s, u, v, k, p and w are Hermitian, $N \times N$ matrices. All of them with exception of u are diagonal. If in addition we suppose that z' lie in $\rho'(\text{centre}\mathcal{A}')$ then we have $r = s = \lambda_q 1_N$, $u = v = \lambda_l 1_N$ with positive constants λ_q, λ_l and we have $k = p = w = 0$. Note that a general z' in the commutant does not commute with the chirality operator unless we set $k = p = w = 0$.

4. Connes–Lott model building kit, space–time added

In this section, the Higgses H are promoted to genuine fields, i.e. space–time dependent vectors. As already in classical quantum mechanics, this promotion is achieved by tensorizing with functions. Let us denote by $\mathcal{F}$ the algebra of (smooth, real or complex valued) functions over space–time M. Consider the algebra $\mathcal{A}_t := \mathcal{F} \otimes \mathcal{A}$. The group of unitaries of the tensor algebra $\mathcal{A}_t$ is the gauged version of the group of unitaries of the internal algebra $\mathcal{A}$, i.e. the group of functions from space–time into the group G. Consider the representation $\rho_t := \underline{\cdot} \otimes \rho$ of the tensor algebra on the tensor product $\mathcal{H}_t := \mathcal{S} \otimes \mathcal{H}$, where $\mathcal{S}$ is the Hilbert space of square integrable spinors on which functions act by multiplication: $(\underline{f\psi})(x) := f(x)\psi(x)$, $f \in \mathcal{F}$, $\psi \in \mathcal{S}$. We denote the genuine Dirac operator by $\partial\!\!\!/$ and its chirality operator by γ_5. The definition of the tensor product of Dirac operators,

$$\mathcal{D}_t := \partial\!\!\!/ \otimes 1 + \gamma_5 \otimes \mathcal{D}, \tag{64}$$

comes from non-commutative geometry. We now repeat the above construction for the infinite-dimensional algebra $\mathcal{A}_t$ with representation ρ_t and Dirac operator $\mathcal{D}_t$. As already stated, for $\mathcal{A} = \mathbb{C}$, $\mathcal{H} = \mathbb{C}$, $\mathcal{M} = 0$, the differential algebra $\Omega_{\mathcal{D}_t}\mathcal{A}_t$ is isomorphic to the de Rham algebra of differential forms $\Omega(M, \mathbb{C})$. For general $\mathcal{A}$, using the notations of [14], an anti-Hermitian 1-form

$$H_t \in \Omega^1_{\mathcal{D}_t}\mathcal{A}_t, \quad H_t^* = -H_t,$$

contains two pieces, an anti-Hermitian Higgs *field* $H \in \Omega^0(M, \Omega^1_{\mathcal{D}}\mathcal{A})$ and a genuine gauge field $A \in \Omega^1(M, \rho(\mathfrak{g}))$ with values in the Lie algebra of the group of unitaries

$$\mathfrak{g} := \{X \in \mathcal{A},\ X^* + X = 0\} \tag{65}$$

represented on $\mathcal{H}$. The curvature of H_t

$$C_t := \delta_t H_t + H_t^2 \ \in \Omega^2_{\mathcal{D}_t}\mathcal{A}_t \tag{66}$$

contains three pieces,

$$C_t = C + F - \mathrm{D}\Phi\gamma_5, \tag{67}$$

the ordinary, now x-dependent curvature $C = \delta H + H^2$, the field strength

$$F := \mathrm{d}A + \tfrac{1}{2}[A, A] \quad \in \Omega^2(M, \rho(\mathfrak{g})) \tag{68}$$

and the covariant derivative of Φ

$$\mathrm{D}\Phi = \mathrm{d}\Phi + [A\Phi - \Phi A] \quad \in \Omega^1(M, \Omega^1_{\mathcal{D}}\mathcal{A}). \tag{69}$$

Note that the covariant derivative may be applied to Φ, thanks to its homogeneous transformation law, Eq. (43).

The definition of the Higgs potential in the infinite-dimensional space

$$V_{\mathrm{t}}(H_{\mathrm{t}}) := (C_{\mathrm{t}}, C_{\mathrm{t}}) \tag{70}$$

requires a suitable regularization of the sum of eigenvalues over the space of spinors $\mathcal{S}$. Here we have to suppose space–time to be compact and Euclidean. Then, the regularization is achieved by the Dixmier trace which allows an explicit computation of V_{t}. One of the miracles in the Connes–Lott scheme is that V_t alone reproduces the complete bosonic action of a Yang–Mills–Higgs model. Indeed, it consists of three pieces, the Yang–Mills action, the covariant Klein–Gordon action and an integrated Higgs potential

$$V_{\mathrm{t}}(A + H) = \int_M \mathrm{tr}(F * F\, z) + \int_M \mathrm{tr}(\mathrm{D}\Phi^* * \mathrm{D}\Phi\, z) + \int_M *V(H). \tag{71}$$

As the preliminary Higgs potential V_0, the (final) Higgs potential V is calculated as a function of the fermion masses,

$$V := V_0 - \mathrm{tr}[\alpha C^* \alpha C\, z] = \mathrm{tr}[(C - \alpha C)^*(C - \alpha C)\, z], \tag{72}$$

where the linear map

$$\alpha : \Omega^2_{\mathcal{D}}\mathcal{A} \longrightarrow \rho(\mathcal{A}) + \pi(\delta(\ker \pi)^1) \tag{73}$$

is determined by the two equations

$$\mathrm{tr}[R^*(C - \alpha C)\, z] = 0 \quad \text{for all } R \in \rho(\mathcal{A}), \tag{74}$$

$$\mathrm{tr}[K^* \alpha C\, z] = 0 \quad \text{for all } K \in \pi(\delta(\ker \pi)^1). \tag{75}$$

All remaining traces are over the finite-dimensional Hilbert space $\mathcal{H}$. We denote the Hodge star by $*$. It should not be confused with the involution *. Note the 'wrong' relative sign of the third term in Eq. (71). The sign is in fact correct for a Euclidean space–time.

A similar miracle happens in the fermionic sector, where the completely covariant action $\psi^*(\mathcal{D}_{\mathrm{t}} + \mathrm{i}H_{\mathrm{t}})\psi$ reproduces the complete fermionic action of a Yang–Mills–Higgs model. We denote by

$$\psi = \psi_{\mathrm{L}} + \psi_{\mathrm{R}} \in \mathcal{H}_{\mathrm{t}} = \mathcal{S} \otimes (\mathcal{H}_{\mathrm{L}} \oplus \mathcal{H}_{\mathrm{R}}),$$

the multiplets of spinors and by ψ^* the dual of ψ with respect to the scalar product of the concerned Hilbert space. For the purpose of this general section, we set

$$H =: \mathrm{i}\begin{pmatrix} 0 & \tilde{h} \\ \tilde{h}^* & 0 \end{pmatrix} \in \Omega^1_{\mathcal{D}}\mathcal{A}, \tag{76}$$

$$\Phi = H - \mathrm{i}\mathcal{D} =: \mathrm{i}\begin{pmatrix} 0 & \tilde{\varphi} \\ \tilde{\varphi}^* & 0 \end{pmatrix} \in \Omega^1_{\mathcal{D}}\mathcal{A}. \tag{77}$$

Then

$$\begin{aligned} \psi^*(\mathcal{D}_\mathrm{t} + \mathrm{i}H_\mathrm{t})\psi &= \int_M *\psi^*(\not\partial + \mathrm{i}\gamma(A))\psi - \int_M *(\psi_\mathrm{L}^*\tilde{h}\gamma_5\psi_\mathrm{R} + \psi_\mathrm{R}^*\tilde{h}^*\gamma_5\psi_\mathrm{L}) \\ &\quad + \int_M *(\psi_\mathrm{L}^*\mathcal{M}\gamma_5\psi_\mathrm{R} + \psi_\mathrm{R}^*\mathcal{M}^*\gamma_5\psi_\mathrm{L}) \\ &= \int_M *\psi^*(\not\partial + \mathrm{i}\gamma(A))\psi - \int_M *(\psi_\mathrm{L}^*\tilde{\varphi}\gamma_5\psi_\mathrm{R} + \psi_\mathrm{R}^*\tilde{\varphi}^*\gamma_5\psi_\mathrm{L}) \end{aligned} \tag{78}$$

containing the ordinary Dirac action and the Yukawa couplings. Note the unusual appearance of γ_5 in the fermionic action (78). Just as the 'wrong' signs in the bosonic action (71), these γ_5 are proper to the Euclidean signature and disappear in the Minkowski signature. For details see the last reference of [1], example 2, 'massless chiral electrodynamics'.

We end this section with a word of caution. In fact, we have slightly over-simplified the outline of the Connes–Lott scheme. The omitted details can be found in [7]. They are irrelevant for our present purpose, the standard model to which we return now.

5. Standard model, space–time added

Let us apply the construction outlined above to the standard model. Recall the expression of the curvature in the electro-weak sector

$$C = (1 - |\varphi|^2)\begin{pmatrix} 1_2 \otimes \Sigma & 0 \\ 0 & \mathcal{M}^*\mathcal{M} \end{pmatrix}. \tag{79}$$

A straightforward application of Eqs. (74) and (75) – taking the real part of the traces is understood – yields the projection αC. It is again block diagonal with diagonal elements:

$$\alpha C_{q\mathrm{L}} = \frac{1 - |\varphi|^2}{2}\,\frac{\mathrm{tr}[M_u^* M_u]\,\mathrm{tr}\,x + \mathrm{tr}[M_d^* M_d]\,\mathrm{tr}\,x + \mathrm{tr}[M_e^* M_e\, y]}{N\,\mathrm{tr}\,x + \mathrm{tr}\,y}\,1_2 \otimes 1_N \otimes 1_3, \tag{80}$$

$$\alpha C_{\ell\mathrm{L}} = \frac{1 - |\varphi|^2}{2}\,\frac{\mathrm{tr}[M_u^* M_u]\,\mathrm{tr}\,x + \mathrm{tr}[M_d^* M_d]\,\mathrm{tr}\,x + \mathrm{tr}[M_e^* M_e\, y]}{N\,\mathrm{tr}\,x + \mathrm{tr}\,y}\,1_2 \otimes 1_N, \tag{81}$$

$$\alpha C_{q\mathrm{R}} = \frac{1 - |\varphi|^2}{2}\,\frac{\mathrm{tr}[M_u^* M_u]\,\mathrm{tr}\,x + \mathrm{tr}[M_d^* M_d]\,\mathrm{tr}\,x + \mathrm{tr}[M_e^* M_e\, y]}{N\,\mathrm{tr}\,x + \mathrm{tr}\,y/2}\,1_2 \otimes 1_N \otimes 1_3, \tag{82}$$

$$\alpha C_{\ell\mathrm{R}} = \frac{1 - |\varphi|^2}{2}\,\frac{\mathrm{tr}[M_u^* M_u]\,\mathrm{tr}\,x + \mathrm{tr}[M_d^* M_d]\,\mathrm{tr}\,x + \mathrm{tr}[M_e^* M_e\, y]}{N\,\mathrm{tr}\,x + \mathrm{tr}\,y/2}\,1_N. \tag{83}$$

The Higgs potential is computed next,

$$V = K(1-|\varphi|^2)^2, \tag{84}$$

$$\begin{aligned} K :=\ & \tfrac{3}{2}\,\mathrm{tr}[(M_u^*M_u)^2]\,\mathrm{tr}\,x + \tfrac{3}{2}\,\mathrm{tr}[(M_d^*M_d)^2]\,\mathrm{tr}\,x + \mathrm{tr}[M_uM_u^*M_dM_d^*]\,\mathrm{tr}\,x \\ & + \tfrac{3}{2}\,\mathrm{tr}[M_e^*M_eM_e^*M_e\,y] - \tfrac{1}{2}L^2\left[\frac{1}{N\,\mathrm{tr}\,x+\mathrm{tr}\,y} + \frac{1}{N\,\mathrm{tr}\,x+\mathrm{tr}\,y/2}\right], \end{aligned} \tag{85}$$

$$L := \mathrm{tr}[M_u^*M_u]\,\mathrm{tr}\,x + \mathrm{tr}[M_d^*M_d]\,\mathrm{tr}\,x + \mathrm{tr}[M_e^*M_e\,y]. \tag{86}$$

Note that the scalar fields φ_1 and φ_2 are not properly normalized, they are dimensionless. To get their normalization straight we have to compute the factor in front of the kinetic term in the Klein–Gordon action:

$$\mathrm{tr}(\,\mathrm{d}\Phi^* * \mathrm{d}\Phi\, z) = *2L|\partial\varphi|^2. \tag{87}$$

Likewise, we need the normalization of the electro-weak gauge bosons:

$$\mathrm{tr}(F * F\, z) = *(N\,\mathrm{tr}\,x + \mathrm{tr}\,y)(\partial_\mu W_\nu^+\,\partial^\mu W^{-\nu} - \cdots). \tag{88}$$

We end up with the following mass relations:

$$m_{\mathrm{W}}^2 = \frac{L}{N\,\mathrm{tr}\,x + \mathrm{tr}\,y}, \tag{89}$$

$$m_{\mathrm{H}}^2 = \frac{2K}{L}. \tag{90}$$

Finally, we turn to the relations among coupling constants. They are due to the fact that the gauge invariant scalar product on the internal Lie algebra, the Lie algebra of the group of unitaries $\mathfrak{g} := \{X \in \mathcal{A},\ X^* + X = 0\}$, in the Yang–Mills action (71) is not general but stems from the trace over the fermion representation ρ on $\mathcal{H}$. Since this representation is faithful the scalar product (31) indeed induces an invariant scalar product on $\mathfrak{g}$.

The fact that the standard model can be written in the setting of non-commutative geometry depends crucially, at this point, on two happy circumstances. Firstly, the electric charge 'generator'

$$Q = \begin{pmatrix} \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix}\otimes 1_N \otimes 1_3 & 0 & 0 & 0 \\ 0 & \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\otimes 1_N & 0 & 0 \\ 0 & 0 & \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix}\otimes 1_N\otimes 1_3 & 0 \\ 0 & 0 & 0 & -1_N \end{pmatrix} \tag{91}$$

is an element of $\mathrm{i}\rho(\mathfrak{g}) \oplus \mathrm{i}\rho'(\mathfrak{g}')$. Indeed it is a linear combination of weak isospin I_3 and elements of the three $u(1)$ factors:

$$Q = \rho\left(\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, 0\right) + \frac{1}{2\mathrm{i}}\,\rho\,(0,\mathrm{i}) + \frac{1}{6\mathrm{i}}\,\rho'(\mathrm{i}1_3, 0) - \frac{1}{2\mathrm{i}}\,\rho'(0,\mathrm{i}). \tag{92}$$

We have put 'generator' in quotation marks because $\mathrm{i}Q$ is a Lie algebra element, not Q. The weak angle θ_{w} measures the proportion of weak isospin in the electric charge:

$$\frac{Q}{|Q|} = \sin\theta_{\mathrm{w}}\,\frac{I_3}{|I_3|} + \cos\theta_{\mathrm{w}}\,\frac{Y}{|Y|}. \tag{93}$$

The hypercharge Y is a linear combination of the three $u(1)$ factors

$$Y := \frac{1}{2\mathrm{i}}\,\rho(0,\mathrm{i}) + \frac{1}{6\mathrm{i}}\,\rho'(\mathrm{i}1_3,0) - \frac{1}{2\mathrm{i}}\,\rho'(0,\mathrm{i}). \tag{94}$$

Here comes the second happy circumstance, this particular combination Y is singled out by two unimodularity conditions. They reduce the group of unitaries $SU(2)\times U(1)\times U(3)\times U(1)$ to $SU(2)\times U(1)\times SU(3)$ with the surviving $U(1)$ generated by the hypercharge. Indeed, the centre of $\mathcal{A}\oplus\mathcal{A}'$ is four dimensional with basis $p_1,\ldots,p_4$. $p_1 := \rho(1_2,0)$ projects on $\mathbb{H}$, $p_2 := \rho(0,1)$ on $\mathbb{C}$, $p_3 := \rho'(1_3,0)$ on $M_3(\mathbb{C})$, and $p_4 = \rho'(0,1)$ on $\mathbb{C}'$, and the group of the standard model is $G_{p_1}\cap G_{p_2}$.

Let us come back to the calculation of the weak angle. Eq. (93) is a matrix of equations. Let us take the difference of the two diagonal elements corresponding to the left-handed neutrino and electron:

$$\frac{1}{|Q|} = \sin\theta_{\mathrm{w}}\,\frac{1}{|I_3|}, \tag{95}$$

$$\sin^2\theta_{\mathrm{w}} = \frac{(I_3,I_3)}{(Q,Q)}. \tag{96}$$

The numerator is readily computed,

$$(I_3,I_3) = \operatorname{tr}\left[\rho\left(\begin{pmatrix}1/2 & 0\\ 0 & -1/2\end{pmatrix},0\right)^2 z\right] = \tfrac{1}{2}(N\operatorname{tr}x + \operatorname{tr}y). \tag{97}$$

We compute the denominator with Pythagoras' kind help,

$$\begin{aligned}(Q,Q) &= \operatorname{tr}\left[\rho\left(\begin{pmatrix}1/2 & 0\\ 0 & -1/2\end{pmatrix},0\right)^2 z\right] + \tfrac{1}{4}\operatorname{tr}[\rho(0,1)^2 z]\\ &\quad + \tfrac{1}{36}\operatorname{tr}[\rho'(1_3,0)^2 z'] + \tfrac{1}{4}\operatorname{tr}[\rho'(0,1)^2 z']\\[1ex] &= (N\operatorname{tr}x + \tfrac{3}{4}\operatorname{tr}y) + \tfrac{1}{6}(\operatorname{tr}r + \operatorname{tr}s) + \tfrac{1}{2}(\operatorname{tr}u/2 + \operatorname{tr}v).\end{aligned} \tag{98}$$

Finally the mixing angle is given by

$$\sin^2\theta_{\mathrm{w}} = \frac{N\operatorname{tr}x + \operatorname{tr}y}{2N\operatorname{tr}x + (3/2)\operatorname{tr}y + (1/3)(\operatorname{tr}r + \operatorname{tr}s) + (1/2)\operatorname{tr}u + \operatorname{tr}v}. \tag{99}$$

In a similar fashion, the ratio between strong and weak coupling is computed,

$$\left(\frac{g_3}{g_2}\right)^2 = \frac{(I_3,I_3)}{(C,C)} = \frac{1}{2}\,\frac{N\operatorname{tr}x + \operatorname{tr}y}{\operatorname{tr}r + \operatorname{tr}s}, \tag{100}$$

where

$$C := \rho'\left(\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0\right). \tag{101}$$

Here C stands for colour not for curvature.

In this calculation z and z' are different in general, implying that the electro-weak sector $\rho(\mathcal{A})$ is orthogonal to the strong sector $\rho'(\mathcal{A}')$. In the special case where $z = z'$ a different choice is possible:

$$(a, a') := \operatorname{tr}[\rho(a)^* \rho'(a')\, z], \quad a \in \mathcal{A},\ a' \in \mathcal{A}'. \tag{102}$$

Then the two $U(1)$ factors $\rho(0, 1)$ and $\rho'(0, 1)$ are not orthogonal anymore and the value of $\sin^2\theta_{\rm w}$ comes out smaller [3]. This choice is closer to grand unified models and yields $\sin^2\theta_{\rm w} = \frac{3}{8} = 0.375$ for $z = z' = 1$ to be compared to $\sin^2\theta_{\rm w} = \frac{12}{29} = 0.414$ from Eq. (99).

6. Conclusions

Writing the standard model in terms of non-commutative geometry yields the four constraints (89, 90, 99, 100) for the W- and Higgs masses, the weak mixing angle and the ratio of strong and weak coupling constants. Note that the off diagonal, chirality mixing terms k, p, and w in z' have dropped out. Due to the highly reducible form of the standard model, these four constraints involve, in the most general case, five arbitrary, positive parameters, the three eigenvalues $y_1,\ y_2,\ y_3$ of the diagonal matrix $3y/\operatorname{tr} x$,

$$\alpha := \frac{\operatorname{tr} r + \operatorname{tr} s}{\operatorname{tr} x}, \tag{103}$$

and

$$\beta := \frac{\operatorname{tr} u/2 + \operatorname{tr} v}{\operatorname{tr} x}. \tag{104}$$

With these parameters the first constraint reads

$$\begin{aligned} m_{\rm t}^2 &= 3m_{\rm W}^2 - (m_b^2 + m_c^2 + m_s^2 + m_d^2 + m_u^2) \\ &\quad + \tfrac{1}{3}y_1(m_{\rm W}^2 - m_e^2) + \tfrac{1}{3}y_2(m_{\rm W}^2 - m_\mu^2) + \tfrac{1}{3}y_3(m_{\rm W}^2 - m_\tau^2) \\ &\approx \left(3 + \frac{y_1 + y_2 + y_3}{3}\right) m_{\rm W}^2. \end{aligned} \tag{105}$$

This approximation is as good as the present day experimental accuracy in the measurement of the W-mass,

$$m_{\rm W} = 80.20 \pm 0.26\ \text{GeV}, \quad \sqrt{m_{\rm W}^2 - 1/3\, m_b^2} = 80.16\ \text{GeV}. \tag{106}$$

For all practical purposes, we therefore have the inequality

$$m_{\rm t} > \sqrt{3}\, m_{\rm W} > \sqrt{3}\, m_e. \tag{107}$$

Similarly we get from the second constraint

$$\sqrt{7/3} < \frac{m_{\rm H}}{m_{\rm t}} < \sqrt{3}. \tag{108}$$

Both constraints together determine the Higgs mass as a function of the top mass:

$$m_{\rm H} \approx \sqrt{11 + 3R - \frac{8+2R}{7+R}}\, m_{\rm W} \tag{109}$$

with

$$R := \frac{m_{\rm t}^2 - 4m_{\rm W}^2}{m_{\rm W}^2} > -1. \tag{110}$$

R vanishes in the subclass of scalar products coming from the centre. Note that the Higgs mass is an increasing function of the top mass, while the renormalization group analysis yields a slowly decreasing function [11].

The third constraint,

$$\sin^2\theta_{\rm w} = \frac{3 + (1/3)\sum y_j}{6 + (1/2)\sum y_j + (1/3)\alpha + \beta}, \tag{111}$$

yields an inequality,

$$\sin^2\theta_{\rm w} < \frac{2}{3}\frac{4+R}{5+R}. \tag{112}$$

The last constraint,

$$\left(\frac{g_3}{g_2}\right)^2 = \frac{3 + (1/3)\sum y_j}{2\alpha}, \tag{113}$$

is empty.

If we take the natural subclass of scalar products with z and z' in the centres, the constraints are more stringent. Indeed, we are now left with only two positive parameters q and ℓ:

$$y_1 = y_2 = y_3 = \frac{\lambda}{\lambda\, {\rm tr}\, 1_3/3} = 1, \tag{114}$$

$$\alpha = \frac{2\lambda_q\, {\rm tr}\, 1_N}{\lambda\, {\rm tr}\, 1_3} = 2\frac{\lambda_q}{\lambda} =: 2q, \tag{115}$$

$$\beta = \frac{(3/2)\lambda_\ell\, {\rm tr}\, 1_N}{\lambda\, {\rm tr}\, 1_3} = \frac{3}{2}\frac{\lambda_\ell}{\lambda} =: \frac{3}{2}\ell, \tag{116}$$

and the constraints read

$$m_{\rm t} = 2\, m_W, \quad R = 0, \tag{117}$$

$$m_{\rm H} = 3.14\, m_W, \tag{118}$$

$$\sin^2\theta_{\rm w} = \frac{4}{(15/2) + (2/3)q + (3/2)\ell} < \frac{8}{15} = 0.533, \tag{119}$$

$$\left(\frac{g_3}{g_2}\right)^2 = \frac{1}{q}. \tag{120}$$

The simplest scalar product is obtained from $z = \rho(1_2, 1)$ and $z' = \rho'(1_3, 1)$. Then $q = \ell$ and we get

$$\sin^2\theta_{\mathrm{w}} = \frac{12}{29} = 0.414, \quad \left(\frac{g_3}{g_2}\right)^2 = 1. \tag{121}$$

We should point out that the most clear cut 'prediction' of Connes and Lott concerns the mass ratio of the W and the Z, a unit ρ-parameter without any appeal. However, even without such numerical tests, it seems clear to us that non-commutative geometry is intrinsic to the standard model. One may very well formulate and test general relativity using flat geometry exclusively. Still, we all agree that Riemannian geometry is the natural setting – for at least two reasons independent of personal taste. We appreciate the use of the powerful computational tools, that the mathematicians have developed in Riemannian geometry. Secondly, there are infinitely more gravitational theories within Euclidean geometry. Likewise, commutative geometry is perfectly sufficient to write down the standard model and to compute cross sections, still non-commutative geometry is superior [7]. Maybe one day, we will know the masses of the top and the Higgs. And maybe then, the elements z and z', that do not come from the centres, will acquire the status of the cosmological constant.

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